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Identification of Nonlinear Nonautonomous State Space Systems from Input-Output Measurements

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Abstract—**This paper presents a method to determine a nonlinear state space model from a finite number of measurements of the inputs and outputs. The method is based on embedding theory for nonlinear systems, and can be viewed as an extension of the subspace identification method for linear systems. The paper describes the underlying theory and provides some guidelines for using the method in practice. To illustrate the use of the identification method, it was applied to a second-order nonlinear system.**

I. Introduction

Real-life systems almost always show nonlinear dynamical behavior. This behavior complicates the task of finding models that accurately describe these systems. While in a large number of applications a linear model shows already satisfactory results, there are numerous situations where linear models are not accurate enough; especially when we deal with very complex systems or require very high performance. Physical knowledge of the system can be a great aid in finding a nonlinear model. However, this knowledge is not always available. In these cases we have to determine a model from a finite number of measurements of the system's inputs and outputs. This approach to nonlinear system modeling is often referred to as *nonlinear black-box identification*. For an overview see [1]. Usually, a nonlinear mapping is fitted from a number of delayed inputs and outputs to the current output. This results in a nonlinear input-output model of the system. This approach, however, neglects the fact that the dynamics of the system are described by a finite dimensional state space. A model taking the state of the system into account can be beneficial, because of the following reasons:

- The state completely describes the dynamics of the system; it is a compact representation of the dynamics. Hence, analyzing the dynamic behavior of the system reduces to analyzing the state.
- When we deal with systems having several inputs and outputs, the state space representation results in a model with fewer degrees of freedom. This results in a better generalization capability.
- Many nonlinear controller design methods are based on a state space representation of the system.

In this paper we present a method to determine a nonlinear state space model from a finite number of measurements of the inputs and outputs. The method was inspired by the subspace identification method for linear systems [2], [3], and is based on embedding theory for nonlinear systems [4], [5].

The paper is organized as follows. Section II gives a short summary of the embedding theory for nonlinear nonautonomous systems. Section III summarizes the subspace identification method for linear systems. Based on the results presented in these sections a method for identification of nonlinear state space systems is presented in section IV. Section V provides some guidelines for choosing the dimension and delay parameters in the identification method. The presented techniques are illustrated in section VI for a simple example system.

II. Embedding of Nonautonomous Dynamics

Let X be a compact manifold of dimension n. Consider the following nonlinear state space system

$$x_{k+1} = f(x_k, u_k) \tag{1}$$
$$y_k = h(x_k) \tag{2}$$

where $f : X \times \mathbb{R}^m \to X$ is a smooth diffeomorphism and $h : X \to \mathbb{R}$ a smooth function. Note that a smooth map f is a diffeomorphism if it is one-to-one and onto and if the inverse map f^{-1} is also smooth. Given only measurements of y_k and u_k, we want to determine a state space system having the same input-output dynamic behavior as the original system (1)–(2). It is important to realize that a unique solution to this identification problem does not exist. This is due to the fact that the state x_k can only be determined up to an embedding $\Psi : X \to M$, with M an $l \geq n$ dimensional compact manifold. Recall that an embedding Ψ on a compact manifold X is a smooth diffeomorphism having a one-to-one derivative map $D\Psi$. In other words, an embedding is a map that does not collapse points or tangent directions [4]. The embedding Ψ can in fact be regarded as a nonlinear state transformation. It corresponds to a change of local coordinates to represent the manifold X.

Let $\xi_k = \Psi(x_k)$. The following state space system has the same input-output dynamic behavior as the system (1)–(2).

$$\begin{aligned} \xi_{k+1} &= \Psi\Big(f\big(\Psi^{-1}(\xi_k), u_k\big)\Big) &&= \bar{f}(\xi_k, u_k) \\ y_k &= h\big(\Psi^{-1}(\xi_k)\big) &&= \bar{h}(\xi_k) \end{aligned}$$

It is important to note that the embedding Ψ does not need to be a square map. Hence, the dimension of the state vector ξ_k can be larger than the dimension of x_k.

Define the map $\Phi : X \to Z$ as follows

$$\begin{aligned} \Phi_{f,h,\mu_k}(x_k) &:= \Big[h(x_k), h(f(x_k, u_k)), \ldots \\ &\qquad h(f(\cdots f(f(x_k, u_k), u_{k+1}) \cdots), u_{k+d-2})\Big]^T \\ &= [y_k, y_{k+1}, \ldots, y_{k+d-1}]^T \end{aligned}$$

where Z is a d dimensional compact manifold and $\mu_k := [u_k, u_{k+1}, \ldots, u_{k+d-2}]^T$. It has been shown by Stark *et al.* [5], [6] that under some minor assumptions, for every μ_k the map Φ_{f,h,μ_k} is an embedding, provided that $d \geq 2n + 1$. This result

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is in fact an extension of Takens embedding theorem for autonomous systems [4], [7]. This means that we can reconstruct the dynamics of the system (1)–(2) using only a finite number of delayed input and output measurements. Let the delay vector z_k be defined as follows

$$z_k := [y_k, y_{k+1}, \dots, y_{k+d-1}]^T$$

then we can write

$$\begin{aligned} z_{k+1} &= \Phi_{f,h,\mu_{k+1}}(x_{k+1}) \\ &= \Phi_{f,h,\mu_{k+1}}\Big(f(x_k, u_k)\Big) \\ &= \Phi_{f,h,\mu_{k+1}}\Big(f(\Phi^{-1}_{f,h,\mu_k}(z_k), u_k)\Big) \\ &= F(z_k, u_k, u_{k+1}, \dots, u_{k+d-1}) \end{aligned}$$

where the inverse map Φ^{-1}_{f,h,μ_k} exists, because Φ_{f,h,μ_k} is an embedding. Hence, the dynamic behavior of the system is completely described by the mapping F. However, we do not end up with a state space description of the form (1). This is due to the fact that the embedding Φ_{f,h,μ_k} depends on μ_k. In other words, the nonlinear state transformation depends on a finite number of 'future' inputs. It is easy to see that if we take the last component of F we get a nonlinear ARX type of model describing the system.

$$\begin{aligned} y_{k+d} = F_{k+d-1}(&y_k, y_{k+1}, \dots, y_{k+d-1}, \\ &u_k, u_{k+1}, \dots, u_{k+d-1}) \end{aligned} \tag{3}$$

To arrive at a state space model of the form (1)–(2) we have to remove the dependence of the delay vector z_k on the 'future' inputs μ_k. This is similar to a technique used in subspace identification of linear systems. Before we explain what we mean by this, we take a closer look at the linear subspace identification method.

III. Review of Linear Subspace Identification

Consider an observable linear state space system

$$x_{k+1} = Ax_k + Bu_k \tag{4}$$

$$y_k = Cx_k \tag{5}$$

Subspace identification [2], [3] is a computationally efficient method to determine from input and output measurements a linear state space system up to a similarity transformation; it provides estimates of the matrices $A_T = TAT^{-1}$, $B_T = TB$, and $C_T = CT^{-1}$ where T is a square nonsingular matrix. In a nutshell, subspace identification consists of three steps:

Step 1: Remove the influence of 'future' inputs

We want to reconstruct the state sequence x_k. It is easy to see that the following equation holds

$$z_k = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{d-1} \end{bmatrix}}_{\Gamma_d} x_k + \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ CB & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ CA^{d-2}B & CA^{d-3}B & \dots & CB \end{bmatrix}}_{H_d} \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+d-1} \end{bmatrix}$$

where $d \geq n + 1$. The first part, $\Gamma_d x_k$, is the response of the system from time k to time $k + d - 1$ due to the initial state x_k. The second part is the response due to the 'future' inputs $u_k, u_{k+1}, \dots, u_{k+d-1}$. To reconstruct the state x_k we have to remove the influence of the 'future' inputs. If the Markov parameters of the system are known, and hence the matrix H_d is known, we can simply do this by subtraction: $\tilde{z}_k := z_k - H_d[u_k, u_{k+1}, \dots, u_{k+d-1}]^T = \Gamma_d x_k$. The vector $\tilde{z}_k$ can be viewed as the response of the system due to the initial state x_k with the input switched off. Note that there exists a clever way to remove the influence of the 'future' inputs without the need to know the matrix H_d. This is done by using a linear projection as described in [2] and [3].

Step 2: Reconstruct the state sequence

Let us store the vectors $\tilde{z}_k$ constructed in the first step into a matrix $\tilde{Z}_k = [\tilde{z}_k, \tilde{z}_{k+1}, \dots, \tilde{z}_{k+N}]$ where $N >> d$. By computing a singular value decomposition (SVD) of this matrix, we can reconstruct the state sequence $X_k = [x_k, x_{k+1}, \dots, x_{k+N}]$ up to a linear state transformation T. Let the SVD of $\tilde{Z}_k$ be given by: $\tilde{Z}_k = USV^T$, then the reconstructed state is given by $\hat{X}_k = S^{1/2}V^T = TX_k$. Note that the number of singular values in S determines the dimension of the state vector. In general the dimension of the state vector $\hat{x}_k$ will be less than the dimension of the delay vector $\tilde{z}_k$.

Step 3: Estimate the model

We use the time sequences y_k, u_k, and $\hat{x}_k$ to determine the matrices A_T, B_T, and C_T. It is easy to see from the equations (4) and (5) that this boils down to solving a linear least squares problem.

IV. Identification of Nonlinear State Space Systems

The vectors z_k, $k = 0, 1, 2, \dots$ are points in the reconstructed state space, and thus describe the dynamics of the system. The elements of the delay vector z_k can be viewed as the coordinates which are used to reconstruct the state space. Of course, we do not want the coordinates of this space to depend on the input signal. Therefore, we have to remove the influence of the 'future' inputs $u_k, u_{k+1}, \dots, u_{k+d-1}$. We propose to use the following modified delay vector

$$\begin{aligned} \tilde{z}_k &:= [\tilde{y}_k, \tilde{y}_{k+1}, \dots, \tilde{y}_{k+d-1}]^T \\ &= [h(x_k), h(f(x_k, 0)), \dots \\ &\qquad h(f(\cdots f(f(x_k, 0), 0)\cdots), 0)]^T \\ &= \tilde{\Phi}_{f,h}(x_k) \end{aligned}$$

The vector $\tilde{z}_k$ does not depend on the 'future' inputs. It can be constructed by taking the state x_k as an initial state and then simulate the system for d time steps with the input set to zero.

Note the similarity to the expression $\tilde{z}_k = \Gamma_d x_k$ for a linear state space system (see section III). It is easy to see that the delay vectors $\tilde{z}_k$ satisfy the following dynamic equation

$$\begin{aligned} \tilde{z}_{k+1} &= \tilde{\Phi}_{f,h}(x_{k+1}) \\ &= \tilde{\Phi}_{f,h}\Big(f(x_k, u_k)\Big) \\ &= \tilde{\Phi}_{f,h}\Big(f(\tilde{\Phi}_{f,h}^{-1}(\tilde{z}_k), u_k)\Big) \\ &= \tilde{F}(\tilde{z}_k, u_k) \end{aligned}$$

Note that this equation shows that to make the transition from $\tilde{z}_k$ to $\tilde{z}_{k+1}$ we still need the input u_k. Thus, the reconstruction coordinates are independent of the input, but the influence of u_k on the dynamics is preserved.

Note that in the previous discussion we assumed that the map $\tilde{\Phi}_{f,h}$ is an embedding. At present we have no formal proof of this. The proposed procedure is mainly motivated by its similarity to the linear subspace method. In order for this procedure to work, it is at least necessary that the system $\tilde{x}_{k+1} = f(\tilde{x}_k, 0)$ is observable [8].

We propose the following identification procedure for nonlinear systems:

Step 1: Remove the influence of 'future' inputs

At present we have no elegant solution that is comparable to the projection used in linear subspace identification. This remains a topic for further research. For now, we propose to use a nonlinear input-output type of model to generate the delay vectors $\tilde{z}_k$. As described in section II we have an embedding theory at our disposal to reconstruct the dynamic behavior of a nonautonomous system from a finite number of delayed input and output measurements. Therefore, we can estimate a nonlinear ARX type of model as in equation (3). In this equation the delay between the lagged inputs and outputs equals one. However, the embedding theory holds in fact for any delay. We fit the following nonlinear input-output model to the data:

$$\begin{aligned} y_{k\tau_{uy}} = \; & G(y_{(k-d_y)\tau_{uy}}, y_{(k-d_y+1)\tau_{uy}}, \ldots, y_{(k-1)\tau_{uy}}, \\ & u_{(k-d_u)\tau_{uy}}, u_{(k-d_u+1)\tau_{uy}}, \ldots, u_{(k-1)\tau_{uy}}) \end{aligned} \tag{6}$$

where τ_{uy} is the embedding delay, and d_u and d_y are the embedding dimensions. In practical applications it is important to choose the proper embedding delay, because we are dealing with a finite number of finite precision measurements. If the delay τ_{uy} is too small, there is almost no difference between the elements of the delay vectors, resulting in a poor embedding. If the delay is too large, the elements are almost uncorrelated and the embedding can become very complicated [9]. Although the theory suggests that the embedding dimensions for the input and output are equal, we allow them to be different. This could reduce the number of parameters needed to describe the nonlinear function G, and hence result in a better generalization performance. In section V we point out how to estimate the delay τ_{uy} and the dimensions d_u and d_y from the input-output measurements.

Note that we can use any kind of nonlinear modeling technique [1] to approximate the mapping G in equation (6), as long as it generalizes well for zero inputs.

From the nonlinear model G we can generate the delay vector

$$\zeta_k := [\tilde{y}_{k\tau_{uy}}, \tilde{y}_{(k+1)\tau_{uy}}, \ldots, \tilde{y}_{(k+s-1)\tau_{uy}}]^T$$

by simulating the system G from time instant $k\tau_{uy}$ up to time instant $(k+s-1)\tau_{uy}$ ($s \geq 2n+1$) and switching off the input at time instant $k\tau_{uy}$.

Step 2: Reconstruct the state sequence

The vectors ζ_k describe the dynamics of the system in a coordinate system that is independent of the input sequence (like $\tilde{z}_k$ in section IV). This means that we can use ζ_k as the state of the system that we want to model. In practice however, the optimal embedding dimension and delay to reconstruct the state may be different from the ones used in step 1. Therefore, we define a new vector η_k as follows

$$\eta_k := [\zeta_{k\tau_x}, \zeta_{(k+1)\tau_x}, \ldots, \zeta_{(k+d_x-1)\tau_x}]^T$$

where d_x is the embedding dimension, and τ_x the embedding delay which is an integer multiple of τ_{uy}. This vector equals the state x_k of the original system, up to a nonlinear state transformation.

Step 3: Estimate the model

Now, the following system is dynamically equivalent to the system (1)–(2).

$$\eta_{k+1} = \mathcal{F}(\eta_k, u_k) \tag{7}$$

$$y_k = [1, 0, \ldots, 0]\eta_k \tag{8}$$

The final step is to approximate the nonlinear mapping $\mathcal{F}$. Again, we can in principle use any kind of nonlinear modeling technique to do this.

V. Estimating Embedding Dimensions and Delays

For the procedure outlined in the previous section, we need to determine the embedding delay τ_{uy} and the embedding dimensions d_u, d_y from the input-output data. We also need to determine the embedding delay τ_x and dimension d_x from the generated vectors ζ_k. Note that the embedding theory does not yield a minimum embedding dimension. It only yields a dimension that is sufficiently large to reconstruct the dynamics. As long as the dimension is large enough to reconstruct the dynamics, it holds that the smaller the dimension the better. This is because the number of parameters to describe the functions G and $\mathcal{F}$ will be reduced, usually resulting in a better model.

The dimensions d_u, d_y, that is the number of delayed inputs and outputs can be determined from the input-output data using the method of false nearest neighbors [10], [11], [12] or the method of Lipschitz numbers [12], [13]. Both these methods start with a low embedding dimension and stepwise increase this dimension. Because we are dealing with both input and output data, we have to increase d_u and d_y one by one to cover all the possible combinations of embedding dimensions. The dimension d_x can be determined by applying the method of false nearest neighbors or the method of Lipschitz numbers to the vectors ζ_k.

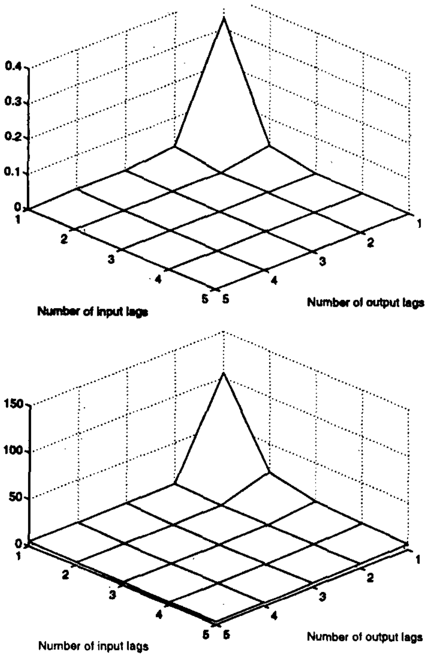

Fig. 1. *Number of false nearest neighbors (top) and Lipschitz numbers (bottom) as function of the number of lagged inputs and outputs.*

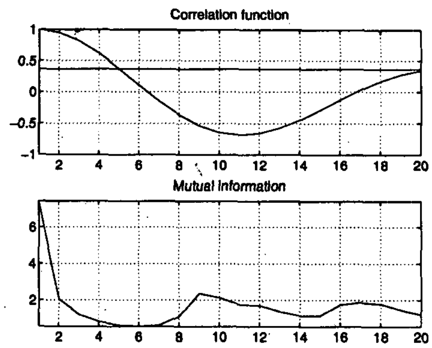

Fig. 2. *Autocorrelation function (top) and mutual information (bottom) of $\tilde{y}_{k\tau_{uy}}$.*

The idea behind the false nearest neighbors method is that if the embedding dimension is too low, there are points in the space that are close together, but will be far apart if the dimension is increased by one. These points are false nearest neighbors, they are not close together because of the dynamics, but due to the fact that the dynamics are projected onto a space that is too small. If the embedding dimension is such that the number of false nearest neighbors is almost zero, the space to reconstruct the dynamics is large enough.

The method of Lipschitz numbers is based on the assumption that the nonlinear function, which we want to approximate, satisfies the Lipschitz conditions. If the embedding dimension is too low, the Lipschitz numbers will be very large while if the dimension is large enough, these numbers will be small.

The embedding delay τ_{uy} can be estimated by evaluating the autocorrelation function or the mutual information [14] of the input and output sequences. The embedding delay τ_x can be estimated using the autocorrelation function or mutual information of $\tilde{y}_{k\tau_{uy}}$. A good estimate of the embedding delay is the value at which the normalized autocorrelation function drops below $1/e$ or the value at which the mutual information obtains its first minimum [9], [14].

VI. Example

In this section, we illustrate the use of the proposed identification method for nonlinear systems for the following example [15]:

$$\begin{aligned}\frac{d}{dt}x_1 &= x_2 - 0.1\cos(x_1)(5x_1 - 4x_1^3 + x_1^5) \\ &\quad -0.5\cos(x_1)u \\ \frac{d}{dt}x_2 &= -65x_1 + 50x_1^3 - 15x_1^5 - x_2 - 100u \\ y &= x_1\end{aligned}$$

This system was simulated using a 4th and 5th order Runge-Kutta method with a sampling time of 0.05 s. The input was a zero-order hold white noise input signal, uniformly distributed between -0.5 and 0.5.

First, we approximated the function G in equation (6). Since, the input is white noise, we took τ_{uy} equal to one. Figure 1 shows the number of false nearest neighbors and the Lipschitz numbers for several combinations of lagged inputs and outputs. From this figure we conclude that the correct values for the embedding dimensions are: $d_u = 2$ and $d_y = 2$. The function G was approximated with a feedforward neural network having one hidden layer that consisted of five neurons. The network was trained using the Levenberg-Marquardt algorithm on 600 data points. Several initial conditions were tried. The network has been validated by looking at the free run performance on a fresh data set.

The network for G was used to generate the vectors ζ_k as described in section IV. To reconstruct the state, we took the embedding delay equal to five ($\tau_x = 5 \cdot \tau_{uy} = 5$), because at lag five the autocorrelation drops below $1/e$, and the mutual information obtains its first minimum (see figure 2). The number of false nearest neighbors and the Lipschitz numbers both indicated an embedding dimension of two. Figure 3 shows the original state trajectories x_k and the reconstructed states η_k.

Finally, we approximated the function $\mathcal{F}$ (see equation (7)) using a neural network with one hidden layer of five neurons. The resulting state space model, described by equations (7)–(8) was validated using a fresh data set. Figure 4 shows the free run simulation results of this model together with the results of a linear state space model, and the results of the nonlinear input-output model used to reconstruct the state. It can been seen that the nonlinear models have a comparable performance, which is much better than the performance of the linear model.

VII. Conclusion

We have presented a method to determine a nonlinear state space model of a system using only measurements of the inputs and outputs. The method consists of three steps: In step one,

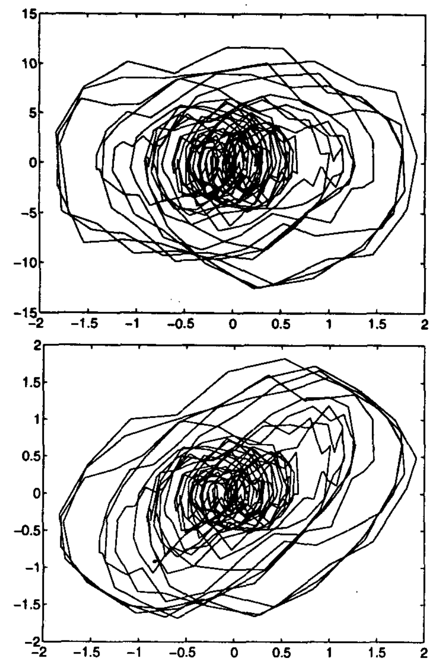

Fig. 3. *Original state trajectories x_k (top) and reconstructed state trajectories η_k (bottom).*

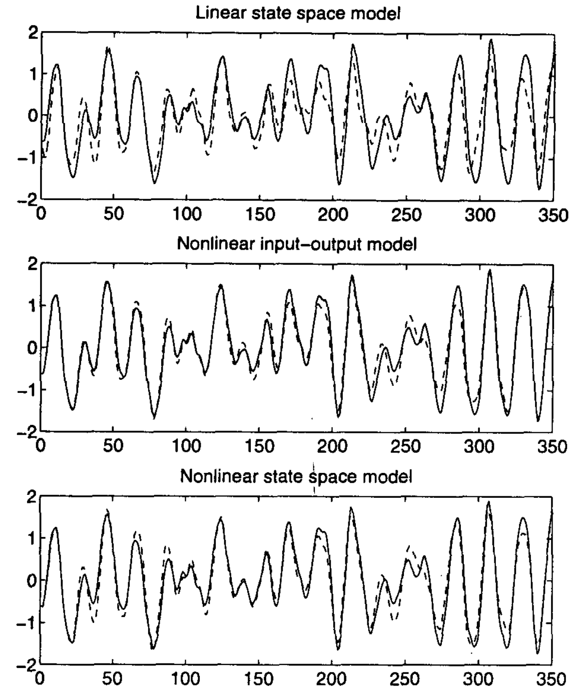

Fig. 4. *Free run simulation of the output. From top to bottom: linear state space model, nonlinear input-output model, nonlinear state space model. The full lines represent the real output, the dashed lines the outputs of the models.*

the influence of 'future' inputs is removed from the 'future' outputs. We use a nonlinear input-output model to do this. In step two, the state sequence of the system is reconstructed up to a nonlinear state transformation. Finally, in step three the model is estimated using the reconstructed state from step two.

Currently, the method can deal with multiple inputs, but not with multiple outputs. Extending the method to multiple outputs is a topic for further research. Another research topic is developing a method to perform step one directly from the data and thus avoiding the use of a input-output model.

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